

More about Latin Tableaux and their Embeddings

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Report CW 668, November 2014



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It is shown that any non-solvable Latin Tableau (LT) has a solvable LT-extension with the same shape, and an algorithm computing the minimal such extension is presented. Minimal Latin Square embeddings of solvable Latin Tableaux are established. The results depend partly on the truth of the Wide Partition Conjecture for Latin Tableaux.

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It is shown that any non-solvable Latin Tableau (LT) has a solvable LT-extension with the same shape. An algorithm computing the minimal such extension is presented. Minimal Latin Square embeddings of classes of solvable Latin Tableaux are established. Embeddings of $LT(N)$ into $LS(N+2)$ are studied in particular. The results depend partly on the truth of the Wide Partition Conjecture for Latin Tableaux.

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1 Introduction

A theorem from [4] implies that every solution of a solvable $LT(N)$ ($N \geq 4$) can be embedded in a solution of a Latin Square of size $2N$. The same paper contains a proof that this bound is tight for partial Latin Squares. [7] shows that this bound is also tight for the smaller class of $LT(N)$: there exist infinitely many $LT(N)$ that have no embedding in smaller than $2N$ -sized Latin Squares. Here we want to study the minimal embeddings (i.e. the smallest possible LS allowing an embedding) for particular Latin Tableaux.

From time to time we need to rely on the Wide Partition Conjecture (WPC) as stated in [2], instead of being able to prove directly that a partition has a solution. So, in Section 2 after an introduction to Latin Tableaux in Section 2, we give a quick introduction to the WPC, and since we use Ryser's theorem a lot, this use is also introduced.

Amongst others, we present (partial) results related to

1. every (unsolvable) $LT(N)$ is the 0-complement of a solvable $LT(M)$,
2. every unsolvable $LT(N)$ can be extended to a solvable $LT(M)$,
3. extending a non-wide partition in a minimal way to a wide partition,
4. the *gap* in embedding an LT into an LS .

With the *gap* we mean the quantity $M - N$ where M is the smallest $M > N$ for which a particular $LT(N)$ has an $LS(M)$ -embedding.

Issues 1-3 are fully solved; issue 4 is solved for particular classes of LT s.

Section 3 shows a minimal same-shape solvable extension for any LT . Section 4 introduces the minimal embedding of (a solution to) an $LT(N)$ in an $LS(M)$: Section 5 deals with the case $M = N + 2$, while Section 6 deals with $M > N + 2$.

2 Preliminaries

2.1 About $LT(N)$

A Latin Tableau can be seen as part of a Latin Square, in which the domain of every cell (i, j) is restricted to $1 \dots \min(\text{size}(\text{row}_i), \text{size}(\text{col}_j))$, and whose form is a *Young diagram*. As an example, Figure 1 shows a solved Latin Tableau of size 8.

An (empty) Latin Tableau is characterized by the sizes of the subsequent rows, so we denote the Tableau in Figure 1 by $LT[8, 7, 7, 5, 5, 3, 3, 1]$. We are mainly interested in studying *square* LT 's, so the length of the list of sizes must be equal to the first number in that list. This list forms a partition of the sum of its elements, so we talk freely about *the partition of an LT* . We use $LT(N)$ to denote any Latin Tableau with N the length of the first row. Note that for convenience, we write the numbers of the defining list in descending order, but the order is really immaterial.

8	6	7	4	5	2	3	1
6	7	4	5	2	3	1	
7	5	6	3	4	1	2	
4	2	5	1	3			
5	4	3	2	1			
2	3	1					
3	1	2					
1							

Figure 1: A solved Latin Tableau

2.2 The Wide Partition Conjecture: a quick intro

What follows in this section is a rephrasing of parts of [2] and [http://en.wikipedia.org/wiki/Partition_\(number_theory\)](http://en.wikipedia.org/wiki/Partition_(number_theory)).

Definition 1. A *partition* (of a natural number I) is a finite non increasing sequence of natural numbers $\mu_i > 0$ (adding up to I).

Example 1. $[7,4,4,1]$ is a partition (of 16) with length 4 □

For convenience we use $X(Y)$ as an abbreviation of Y times the number X in a partition. So, $[4(3),3]$ means the same as $[4,4,4,3]$.

Definition 2. A *Young diagram* is a visual representation of a partition μ : a left alignment of rows so that the i^{th} row has μ_i square cells.

Example 2. Figure 2 shows the Young diagram of $[7,4,4,1]$. □

Figure 2: The Young diagram of $[7,4,4,1]$

For a given (Young diagram of a) partition μ , let us number the rows also from 1 up to the length of the partition. Then each cell can be denoted by its coordinates (i,j) as $cell(i,j)$, for sensible i,j .

Definition 3. The *Young-CSP* (Constraint Satisfaction Problem) associated with a partition $\mu = [\mu_1, \dots, \mu_l]$ has the following three ingredients:

- $Vars = \{cell(i,j) \mid cell(i,j) \in \text{the Young-diagram of } \mu\}$
- $Dom = \{cell(i,j) \rightarrow [1..\mu_i] \mid j \in [1..\mu_i]\}$
- $Constraints = \bigcup_{i=1}^l (\{cell(i,j) \neq cell(i,j') \mid \forall j,j' : j \neq j'\} \cup \{cell(j,i) \neq cell(j',i) \mid \forall j,j' : j \neq j'\})$

Another way to phrase the constraints is: in every row and column, the variables must be *all_different*. [9]

Example 3. The domains and the constraints of the Young-CSP of $[7,4,4,1]$ are

$\text{cell}(1,1), \text{cell}(1,2), \text{cell}(1,3), \text{cell}(1,4), \text{cell}(1,5), \text{cell}(1,6), \text{cell}(1,7) \rightarrow [1..7]$
 $\text{cell}(2,1), \text{cell}(2,2), \text{cell}(2,3), \text{cell}(2,4) \rightarrow [1..4]$
 $\text{cell}(3,1), \text{cell}(3,2), \text{cell}(3,3), \text{cell}(3,4) \rightarrow [1..4]$
 $\text{cell}(4,1) \rightarrow [1..1]$
 $\text{all_different}([\text{cell}(1,1), \text{cell}(1,2), \text{cell}(1,3), \text{cell}(1,4), \text{cell}(1,5), \text{cell}(1,6), \text{cell}(1,7)])$
 $\text{all_different}([\text{cell}(2,1), \text{cell}(2,2), \text{cell}(2,3), \text{cell}(2,4)])$
 $\text{all_different}([\text{cell}(3,1), \text{cell}(3,2), \text{cell}(3,3), \text{cell}(3,4)])$
 $\text{all_different}([\text{cell}(1,1), \text{cell}(2,1), \text{cell}(3,1), \text{cell}(4,1)])$
 $\text{all_different}([\text{cell}(1,2), \text{cell}(2,2), \text{cell}(3,2)])$
 $\text{all_different}([\text{cell}(1,3), \text{cell}(2,3), \text{cell}(3,3)])$
 $\text{all_different}([\text{cell}(1,4), \text{cell}(2,4), \text{cell}(3,4)])$

□

Not every Young-CSP has a solution, but the example above has:

4	3	2	1	5	6	7
3	2	1	4			
2	1	4	3			
1						

Figure 3: Solution to the $[7,4,4,1]$ -CSP

Definition 4. The *conjugate partition* $\text{conj}(\mu)$ of a partition μ is the partition corresponding to the Young diagram obtained by flipping the Young diagram of μ over the main diagonal. If the conjugate of μ equals μ , we say that μ is *self-conjugate*.

Example 4. Figure 4 shows the conjugate Young diagram of $[7,4,4,1]$: the conjugate partition of $[7,4,4,1]$ is $[4,3,3,3,1,1,1]$.

One can check that this conjugate Young diagram has no solution. □

Definition 5. A partition μ *dominates* a partition ν , denoted as $\mu \geq \nu$ if $\forall j, \sum_{i=1}^j \mu_i \geq \sum_{i=1}^j \nu_i$ where any non-existent μ_i or ν_i are taken to be zero.

Example 5. One can check that $[7,4,4,1]$ dominates $[4,3,3,3,1,1,1]$ but not the other way around. □

Definition 6. A *lower subpartition* of a partition $\mu = [\mu_1, \mu_2, \dots, \mu_l]$ is any partition $[\mu_k, \mu_{k+1}, \dots, \mu_l]$ for $k \in 1..l$.

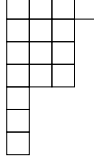


Figure 4: The Young diagram of the conjugate of $[7,4,4,1]$

Example 6. The lower subpartitions of $[4,3,3,3,1,1,1]$ are $[4,3,3,3,1,1,1]$, $[3,3,3,3,1,1,1]$, $[3,3,3,1,1,1]$, $[3,1,1,1,1]$, $[1,1,1,1]$, $[1,1]$, $[1]$. \square

Definition 7. A partition μ is *wide* if for every lower subpartition λ of μ , $\lambda \geq \text{conj}(\lambda)$.

The definition above is not the original one, but the one proven to be equivalent in Proposition 3 of [2].

Example 7. $[7,4,4,1]$ is wide, because

$$\begin{aligned} [7, 4, 4, 1] &\geq [4, 3, 3, 3, 1, 1, 1] \quad \wedge \\ [4, 4, 1] &\geq [3, 2, 2, 2] \quad \wedge \\ [4, 1] &\geq [2, 1, 1, 1] \quad \wedge \\ [1] &\geq [1] \end{aligned} \quad \square$$

For a partition $\mu = [\mu_1, \mu_2, \dots, \mu_l]$, define $\text{chop}(\mu) = [\mu_2, \dots, \mu_l]$.

Alternative definition of *wide* It is easily checked that the following is an equivalent definition of wideness: a partition λ is *wide* if λ is a singleton, or $\lambda \geq \text{conj}(\lambda)$ and if $\text{chop}(\lambda)$ is wide.

Conjecture 1. The **Wide Partition Conjecture** from [2]: A Young-CSP has a solution iff its corresponding partition is wide.

The above is not exactly what [2] states, but it is enough for the purpose of this report. Since we believe the conjecture is true, we use *solvable* and *wide* (almost) as synonyms.

Definition 8. For a partition $\mu = [\mu_1, \mu_2, \dots, \mu_l]$, define $\text{addcol}(\mu) = [\mu_1 + 1, \mu_2 + 1, \dots, \mu_l + 1]$. On the diagram, this corresponds to adding a full column at the left.

Definition 9. For a partition $\mu = [\mu_1, \mu_2, \dots, \mu_l]$, define $\text{addrow}(\mu) = [\mu_1 + 1, \mu_2 + 1, \dots, \mu_l + 1]$. On the diagram, this corresponds to adding a full row on top.

Note that addrow and addcol commute. We denote $\text{addrow} \circ \text{addcol}$ by ext .

Definition 10. The *shape* of a partition $\mu = [\mu_1, \mu_2, \dots, \mu_l]$ is a sequence of numbers defined as follows:

if $(l = 0) \vee (\mu_1 = \mu_l)$ **then** $shape(\mu) = []$
else let i be smallest i such that $\mu_1 \neq \mu_i$, **then** $shape(\mu) = [a_1, b_1 | R]$
with $a_1 = (\mu_1 - \mu_i)$ **and** $b_1 = \#\{k | \mu_i = \mu_k\}$ **and** $R = shape(chop^{i-1}(\mu))$

As an example: the shape of $LT[7,4,4,1]$ is $[3, 2, 3, 1]$. The shape of a partition is invariant under addrow and addcol. One can check that the shape of a self-conjugate partition is a palindrome. The shape of an LT captures the ragged edge of the LT.

Definition 11. The Young diagram for an $LT(N)$ can be put in the upper left corner of a square with size $(N + k)$. Deleting from this square the cells belonging to the $LT(N)$ results in the k -complement of the $LT(N)$.

Clearly, the k -complement of an $LT(N)$ can be considered as an $LT(M)$. If the k -complement has a solution, then by shifting/inverting the domains of its cells, it completes a solution of the $LT(N)$ so that together, they form a solution for $LS(M)$. Figure 5 shows this.

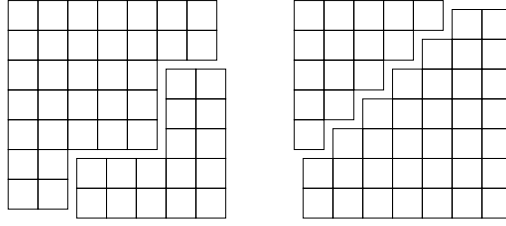


Figure 5: An $LT(7)$ with its 0-complement, and an $LT(5)$ with its 2-complement

Note that in Figure 5, the $LT(7)$ has a solution, but its 0-complement has not. The $LT(5)$ in Figure 5 and its 2-complement both have a *solution*, as shown in Figure 6. Together they form a solution to $LS(7)$.

5	4	3	2	1		7	6
4	3	2	1		7	6	5
3	2	1		7	6	5	4
2	1		7	6	5	4	3
1		7	6	5	4	3	2
	7	6	5	4	3	2	1
	6	5	4	3	2	1	7

Figure 6: A solution of an $LT(5)$ and its 2-complement: together a solved $LS(7)$

2.3 Using Ryser's Theorem

We will use Ryser's Theorem [8] quite often, so we feel it is worthwhile to review it here and explain how we use it:

Theorem by H. Ryser:

Let $N_R(o_i)$ denote the number of occurrences of o_i in an $r \times s$ latin rectangle R on the symbols o_1, \dots, o_n , then R may be completed to form a latin square of side n if and only if $\forall i : 1 \leq i \leq n \rightarrow N_R(o_i) \geq r + s - n$

We use it in the context of completing a solution of an $LT(N)$ to a solution of a Latin Square $LS(M)$, possibly with $M > N$. Consider the 0-complement B of an $LT(N)$ A . B is itself an $LT(NC)$. Fill A with a solution, and suppose we can fill B with the values $1..M$ so that rows and columns of the square $A \cup B$ contain every value at most once: note that not every value of $1..M$ needs to be used for that. Clearly, $M \geq N$. If $M = N$ we are done, but if the $LT(NC)$ has no solution, $M > N$ and this means that we have used $(M - N)$ new values. Let \mathcal{M} denote the minimum of occurrence counts of each value in $1..M$, i.e. $\mathcal{M} = \min_{i=1}^M \{N_R(i)\}$. Then Ryser says that if $\mathcal{M} \geq 2N - M$, there is an $LS(M)$ -embedding.

3 A minimal same-shape wide $LT(M)$ extension for every $LT(N)$

In [7], it was proven¹ that the 0-complement of a *solvable* $LT(N)$ can be extended to a solvable $LT(M)$ with the same shape. We want the same result but without the solvability condition on the $LT(N)$. In fact, this result was proven already in [2], essentially by making the partition *squarish*. Here however, we are interested in the minimal solvable extension and an algorithm to construct it.

For a given partition $\mu = [\mu_1, \mu_2, \dots, \mu_l]$, find the highest i so that $chop^i(\mu)$ does not dominate its conjugate: we name this index the *culprit* of μ : $culprit(\mu)$. If none exists, μ is wide and we define $culprit(\mu) = 0$.

Lemma 1. Let $\mu = [\mu_1, \mu_2, \dots, \mu_l]$. If $chop(\mu)$ is wide, then $addcol(\mu)$ is wide.

Proof. Proposition 4 from [2] implies that $addcol(chop(\mu))$ is wide, so all we need to prove is that $addcol(\mu)$ dominates its conjugate. Let $conj(\mu) = \lambda$. $chop(\mu)$ dominates its conjugate, so $\sum_{i=2}^k \mu_i \geq \sum_{i=1}^{k-1} (\lambda_i - 1)$ or

$$\sum_{i=2}^k (\mu_i + 1) \geq \sum_{i=1}^{k-1} \lambda_i \quad (1)$$

In particular, for $k=1$, this implies $\mu_2 + 1 \geq \lambda_1$ and since $\mu_1 \geq \mu_2$, and $\lambda_1 = l$ we derive

$$\mu_1 + 1 \geq l. \quad (2)$$

Adding (1) and (2) yields

$$\sum_{i=1}^k (\mu_i + 1) \geq (l + \sum_{i=1}^{k-1} \lambda_i)$$

which expresses that $addcol(\mu)$ dominates its conjugate. □

¹but not explicitly stated.

From Lemma 1 we can conclude:

Corollary 1. If $\text{culprit}(\mu) > 0$ then $\text{culprit}(\mu) > \text{culprit}(\text{addcol}(\mu))$, else $\text{culprit}(\text{addcol}(\mu)) = 0$.

Lemma 2. Let $\text{ext}(\mu) = \text{addrow}(\text{addcol}(\mu))$. If μ is self-conjugate and μ is wide, $\text{ext}(\mu)$ dominates its conjugate and is wide.

Proof. From Proposition 4 in [2] we know that $\text{addcol}(\mu)$ is wide. Since μ is self-conjugate, $\text{ext}(\mu)$ is self-conjugate, so the result follows because a self-conjugate partition dominates its conjugate. \square

If the WPC is true, then we can conclude at this point that if an LT(N) μ is solvable, then $\text{ext}(\mu)$ is solvable. It would be nice to have a constructive proof independent of the WPC.

Theorem 3.1. If μ is self-conjugate and $\text{addcol}^k(\mu)$ is wide, then $\text{ext}^k(\mu)$ is wide.

Proof. It is enough to prove that $\forall i : 0 < i \leq k$ $\text{addrow}^i(\text{addcol}^k(\mu))$ dominates its conjugate. For $i = k$ this is trivial, because $\text{ext}^k(\mu)$ is self-conjugate. So, let $0 < i < k$.

Figure 7 visualizes $\mu = [\mu_1, \mu_2, \dots, \mu_l]$, the k added columns and rows, i , and a p that is used later. Corollary 1 implies that the smallest k for which $\text{addcol}^k(\mu)$ is wide, is strictly smaller than l . For larger k we can use Lemma 2, so we make the proof for $k < l$.

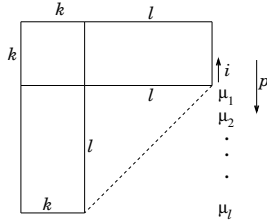


Figure 7: $\text{ext}^k(\mu)$

The figure makes it easy to see that

$\text{addrow}^i(\text{addcol}^k(\mu)) = [(k+l)(i), \mu_1 + k, \mu_2 + k, \dots, \mu_l + k] = [\alpha_1, \dots, \alpha_{i+l}]$
 $\text{conj}(\text{addrow}^i(\text{addcol}^k(\mu))) = [(i+l)(k), \mu_1 + i, \mu_2 + i, \dots, \mu_l + i] = [\beta_1, \dots, \beta_{k+l}]$
and we must prove that for every partial sum of the first p elements: $\sum_{j=1}^p \alpha_j \geq \sum_{j=1}^p \beta_j$. We break it down in four separate cases:

$0 < p \leq i$:

The partial sums are $\sum_{j=1}^p (k+l)$ and $\sum_{j=1}^p (i+l)$. Since $k > i$, the inequality follows.

$i < p \leq k$:

We must prove $\sum_{j=1}^i (k+l) + \sum_{j=1}^{p-i} (\mu_j + k) \geq \sum_{j=1}^p (i+l)$

Since $\text{addcol}^k(\mu)$ is wide, we know $\sum_{j=1}^{p-i} (\mu_j + k) \geq \sum_{j=1}^{p-i} l$, so

$$\begin{aligned} \sum_{j=1}^i (k+l) + \sum_{j=1}^{p-i} (\mu_j + k) &\geq \sum_{j=1}^i (k+l) + \sum_{j=1}^{p-i} l \\ &= \sum_{j=1}^p l + i \times k \geq \sum_{j=1}^p l + i \times p \\ &= \sum_{j=1}^p (i+l) \end{aligned}$$

which we needed to prove.

$k < p \leq i+l$:

We must prove

$$\sum_{j=1}^i (k+l) + \sum_{j=1}^{p-i} (\mu_j + k) \geq \sum_{j=1}^k (i+l) + \sum_{j=1}^{p-k} (\mu_j + i)$$

Add $\sum_{j=1}^{p-k} (k-i)$ to both sides to obtain (in a reordered form):

$$\begin{aligned} \sum_{j=1}^i (k+l) + \sum_{j=1}^{p-k} (k-i) + \sum_{j=p-k+1}^{p-i} (\mu_j + k) + \sum_{j=1}^{p-k} (\mu_j + k) \\ \geq \sum_{j=1}^k (i+l) + \sum_{j=1}^{p-k} (\mu_j + k) \end{aligned}$$

Subtract $\sum_{j=1}^{p-k} (\mu_j + k)$ from both sides to get

$$\sum_{j=1}^i (k+l) + \sum_{j=1}^{p-k} (k-i) + \sum_{j=p-k+1}^{p-i} (\mu_j + k) \geq \sum_{j=1}^k (i+l) \quad (3)$$

Since in $\text{addcol}^k(\mu)$ the partition starting at the $(p-k+1)^{th}$ row dominates its conjugate, we know

$\sum_{j=p-k+1}^{p-i} (\mu_j + k) \geq \sum_{j=p-k+1}^{p-i} (l - (p-k+1) + 1)$, so starting from the lefthand side of (3), we get

$$\begin{aligned} \sum_{j=1}^i (k+l) + \sum_{j=1}^{p-k} (k-i) + \sum_{j=p-k+1}^{p-i} (\mu_j + k) \\ \geq \sum_{j=1}^i (k+l) + \sum_{j=1}^{p-k} (k-i) + \sum_{j=p-k+1}^{p-i} (l-p+k) \\ = i \times (k+l) + (k-i) \times (p-k) + (l-p+k) \times (k-i) = k \times (i+l) \\ = \sum_{j=1}^k (i+l) \end{aligned}$$

and the inequality (3) is proven.

$i+l < p$:

The partial sum now exhausts α and there is some of β left because $i < k$, so it follows immediately.

□

Based on the lemmas and theorem, we state three versions of an algorithm that extends a self-conjugate partition to a wide partition with the same shape.

Algorithm 1 Extending a self-conjugate partition to a wide partition

```

function EXTEND_TO_WIDE( $\mu$ )
  if  $\text{culprit}(\mu) > 0$  then
    return  $\text{extend\_to\_wide}(\text{ext}(\mu))$ 
  else
    return  $\mu$ 
  end if
end function

```

Algorithm 2 Extending a self-conjugate partition to a wide partition

```

function EXTEND_TO_WIDE( $\mu$ )
   $k := 0$ 
  while  $\text{culprit}(\mu) > 0$  do
     $\mu = \text{addcol}(\mu)$ 
     $k = k + 1$ 
  end while
  return  $\text{addrow}^k(\mu)$ 
end function

```

Algorithm 3 Extending a self-conjugate partition to a wide partition: to be called as $\text{extend_to_wide}(\mu, \mu)$

```

function EXTEND_TO_WIDE( $\mu, \lambda$ )
  if  $\text{culprit}(\mu) > 0$  then
    return  $\text{extend\_to\_wide}(\text{addcol}(\mu), \text{ext}(\lambda))$ 
  else
    return  $\lambda$ 
  end if
end function

```

Theorem 3.2. *Any of the Algorithms 1, 2 and 3 applied on a self-conjugate partition μ , returns the smallest solvable self-conjugate extension with the same shape.*

Proof. This follows from the theorems and the lemmas. □

Figure 8 shows the workings of the algorithm.

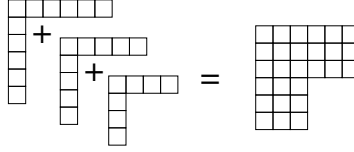


Figure 8: A non-wide LT (the smaller one) and its extension to a wide LT by performing *ext* twice.

Corollary 2 answers a question left open in [7]:

Corollary 2. Every $\text{LT}(N)$ is the 0-complement of a solvable $\text{LT}(M)$

4 The smallest Latin Square embedding of Latin Tableaux

Talking about an LS-embedding of an LT only makes sense if the LT itself is solvable. So, when discussing an LS-embedding of a particular LT, we either assume or (have to) prove that this LT is solvable. Clearly, if an $LT(N)$ has an embedding in an $LS(M)$, $M \geq N$, the $(M - N)$ -complement of the $LT(N)$ is a solvable LT. We often show such complement is solvable by using Ryser's theorem so that we can restrict our attention to the $N \times N$ -square (of which the $LS(N)$ is a part) within the $LS(M)$.

Consider Figure 9: it shows the smallest possible $LT(5)$ (full lines). The smallest possible 3-complement of an $LT(5)$ cannot be smaller than the $LT(8)$ drawn with dotted lines in the same figure. In between are cells (in bold lines) that might belong to a solvable $LT(5)$ or to its solvable 3-complement. Indeed, a solvable $LT(N)$ must at least have the cells of a $DLT(N)^2$, or to put it differently: a solvable $LT(N)$ must have all the cells above (and including) its diagonal.

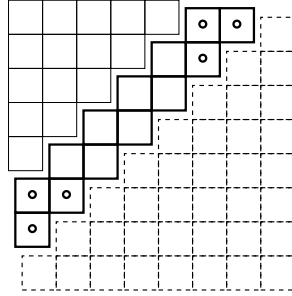


Figure 9: A $DLT(5)$ and a $DLT(8)$; in between the cells that shift the boundary between a solvable $LT(5)$ and its solvable 3-complement

Certainly, the cells with a dot must belong to a 3-complement of an $LS(5)$, but the important thing for us at this point is that there is a stretch of cells, with width equal to two, that could belong to the $LS(5)$ when we want its 3-complement to be solvable, and not more than that.

This observation can be generalized readily: an $LT(N)$ that can be embedded in an $LS(N+i)$ (with $i > 0$) contains the cells of a $DLT(N)$ plus at most the cells in the stretch of width $(i - 1)$.

This observation leads immediately to

Theorem 4.1. *An $LT(N)$ with an $LS(N+1)$ -embedding equals $DLT(N)$.*

Proof. The stretch has width zero, so the $LT(N)$ equals $DLT(N)$, and the 1-complement equals $DLT(N+1)$. \square

²[6] defines the class of *diagonal LT*: $DLT(N) = LT[N, N-1, N-2, \dots, 1]$; a $DLT(N)$ has exactly one solution

An experiment We programmed an exhaustive search of all possible gaps for $LT(N)$ for $N = 1..50$: we relied on the WPC, so no constraint solver was needed, so we programmed this in hProlog. The outcome indicated a surprise:

- (1) For all N and $3 \leq i \leq N$, there is an $LT(N)$ with gap i .
- (2) There is an $LT(N)$ with gap 2 if and only if $(N+2)$ is composite.

The surprise was: we had expected that the gap 2 was present for **all** N . We found a method to construct systematically $LT(N)$ with gap 2 (for composite $(N+2)$), so the *if*-part of the second statement has a proof: see Section 5. We have made little progress on the *only if* part of that statement. The first statement is discussed in Section 6.

5 Latin Tableaux with a 2-gap

We start by showing that for composite $(N+2)$, there is a $LT(N)$ with a gap equal to 2. One alternative (equivalent) way to formulate this is: for composite $(N+2)$ there exists a solvable $LT(N)$ whose 2-complement is solvable. Another way is: for composite $(N+2)$ there exists a solvable $LT(N)$ so that its 0-complement can be filled using the numbers $1..(N+2)$ so that the resulting square of size $N \times N$ fulfills the conditions of Ryser's theorem to embed it in an $LS(N+2)$. We use the latter formulation.

Definition 12. $LTe(N)$ is a solvable $LT[N, x_2, x_3, \dots, x_{N-1}, x_N]$ such that $x_j = (N - j + 1)$ or $x_j = (N - j + 2)$.

In words: an $LTe(N)$ is like $DLT(N)$, but some rows have an extra cell - and it has a solution. A subset of the $LTe(N)$ is defined next:

Definition 13. $LT_d(N)$ is the $LT[N, x_2, x_3, \dots, x_{N-1}, x_N]$ such that $x_j = (N - j + 1) + (1 == (j \bmod d)) ? 1 : 0$

In words: $LT_d(N)$ is like $DLT(N)$, but with an extra cell added to each row that is a multiple of d . It is easy to check that $LT_d(N)$ is self-conjugate if and only if d divides $(N+2)$, meaning that $LT_3(5)$ does not make sense as an LT , but $LT_5(13)$ and $LT_4(6)$ do.

Theorem 5.1. *If d divides $(N+2)$, $LT_d(N)$ is solvable.*

Proof. Figure 10 shows solutions for $LT_5(13)$ and $LT_3(13)$.

Figure 1 displays two triangular grids illustrating the distribution of numbers 1 through 13. The left grid shows a standard distribution where each number appears in a specific row and column. The right grid shows a modified distribution where some numbers are bolded, indicating a change in their position or frequency.

13	12	11	10	9	8	7	6	5	4	3	2	1
12	11	10	9	8	7	6	5	4	3	2	1	
11	10	9	8	7	6	5	4	3	2	1		
10	9	8	7	6	5	4	3	2	1			
9	8	7	6	10	4	3	2	1	5			
8	7	6	5	4	3	2	1					
7	6	5	4	3	2	1						
6	5	4	3	2	1							
5	4	3	2	1								
4	3	2	1	5								
3	2	1										
2	1											
1												

13	12	11	10	9	8	7	6	5	4	3	2	1
12	11	10	9	8	7	6	5	4	3	2	1	
11	10	12	8	7	9	5	4	6	2	1	3	
10	9	8	7	6	5	4	3	2	1			
9	8	7	6	5	4	3	2	1				
8	7	9	5	4	6	2	1	3				
7	6	5	4	3	2	1						
6	5	4	3	2	1							
5	4	6	2	1	3							
4	3	2	1									
3	2	1										
2	1	3										
1												

A number with an arrow means that the corresponding diagonal is filled completely with that number.

Figure 11 shows that the two new numbers (14 and 15) occur 11 times in the 0-complement. The number 15 can occur only 8 times in the main diagonal, but one can recuperate 3 cells in the diagonal for 12. This cascades to the diagonal with 9, and further to the diagonal with 6. Again, this scheme works for any proper divisor of $(N + 2)$.

A formal proof is possible and boring. An example is more informative: Figure 12 shows how to reduce the problem of filling the 0-complement of $LT_3(13)$ to filling the 0-complement of $LT_3(10)$, and further to filling the 0-complement of $LT_3(7)$ and $LT_3(4)$. The X's in the figure are merely for indicating the cells that lie to the right of the main diagonal.

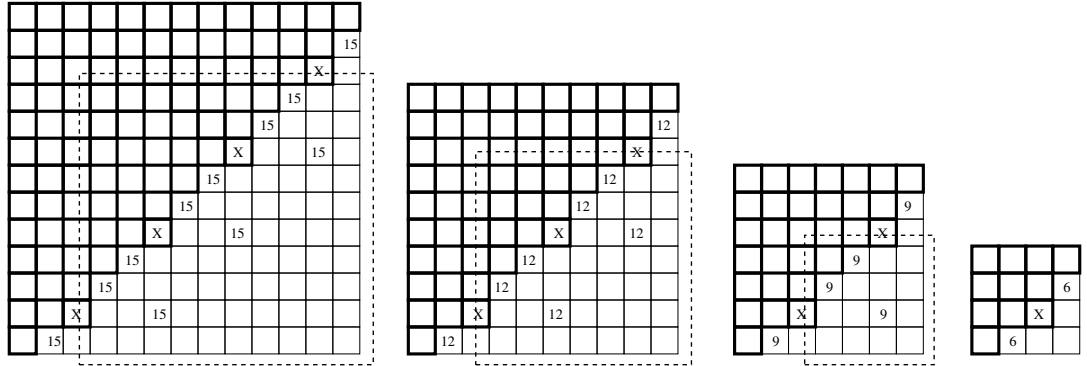


Figure 12: Proof by example

Every reduction step reduces the number of X's by one, so that when there is only one X left, we have a base case, actually an $LT_d(2d - 2)$ which is easily seen to have a gap equal to two.

About $\text{prime}(N+2)$ Empirically, up to $N=50$ or so, we found out that if $(N + 2)$ is prime, then for a particular $LT(N)$ A , if A is wide, then its 2-complement is not (unless $A = DLT(N)$). We need to prove this only for A which differ from $DLT(N)$ on the diagonal next to the main diagonal, as for larger differences, the 2-complement cannot be wide. So we need to establish a connection between the wideness of an $LT(N)$ and the wideness of its 2-complement. In one form, we would like to prove

Conjecture 2. If the $LT(N)$ A is wide, and $(N + 2)$ is prime, then the 2-complement of A is not wide.

We have been unable to prove this.

We observed (for N up to 50 or so) that an $LT(N)$ (with composite $(N + 2)$) has a gap equal to two only if it equals $LT_d(N)$ with d a divisor of $(N + 2)$ (we

have no proof of this yet). This indicates that a proof of Conjecture 2 could follow the following path:

- prove that in the 0-complement of an $LTe(N)$ that satisfies Ryser for a 2-extension, there is a solution in which the first and second diagonal contain only the new values
- the number of cells containing the new values in these diagonals is less than needed, so there must be occurrences of these values in other cells: that can only be in the cells at the intersection of rows/columns in which this value was not present because of the difference between the $LTe(N)$ and $DLT(N)$: prove that in case $(N + 2)$ is prime, the *wrong* values are affected

6 Gaps greater than 2

Ideally, we would like to prove the statement *for all N , there is an $LT(N)$ with gap $i, \forall i \in [3..N]$* : this was established empirically for N up to 50 (or so). We have not worked enough to find a full proof, but examples of $LT(N)$ with particular gaps are shown in the sections to come.

6.1 The gap for $LT[N(i), N-1(N-i-1), i]$ equals $N, N-1$ or $N-2$

Define $LTP[N, i] = LT[N(i), N-1(N-i-1), i]$, for $i = 1..N-1$. Figure 13 shows $LTP[8, 3]$: it also indicates a column C and row R . We number the cells of C top-to-bottom and R left-to-right from 1, so the cell $C[N]$ is the same cell as $R[N]$. We are actually only interested in the parts of C and R that start at index $N+i+1$.

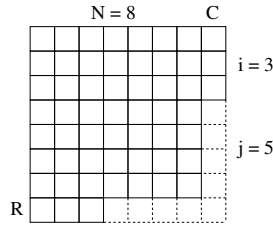


Figure 13: $LTP[8, 3]$

Since an LT can have an embedding in an LS only if it is solvable, the following theorem needs to be proven.

Theorem 6.1. *Every $LTP[N, i]$ has a solution.*

Proof. [1] contains the useful Lemma 2.3.

For any $n \geq 4$, there exists an $n \times n$ latin square, on $1, 2, \dots, n$, which has symbol n in each main diagonal cell, and has the entries of the last row in the same order as the last column.

[illegible]

7 1	7 1	7 1	7 1	7 1
. 7 2	. 7 2	. 7 2	. 7 2	. 2
. . 7 . . . 3	. . 7 . . . 3	. . 7 . . . 3	. . 3 3
. . . 7 . . 4	. . . 7 . . 4	. . . 4 4 4 . . .
. . . . 7 . 5 5 5 5 5 . .
. 6 6 6 6 6
1 2 3 4 5	1 2 3 4	1 2 3	1 2	1

Theorem 6.2. *Let $N = i + j$. The minimal $M > N$ for which $LTP[N, i]$ has an $LS(M)$ -embedding, is for*

- Now suppose that $LTP[N,i]$ has an embedding in $LS(2N-2)$. Then the $N \times N$ -square surrounding the $LTP[N,i]$ must have an assignment (a completion starting from the solution to the $LTP[N,i]$) with $(N-2)$ new numbers that each occur at least twice in the square. Since the number of cells in C and R where such

6.2 LT(N) with gap $N/2$ for even N

Consider $LT[2n(n), n(n)]$: Figure 16 shows an instance for $n = 5$ within the solid line. One can see that $LT[2n(n), n(n)]$ consists of three $LS(n)$ and a solution for each of them is specified by a circle with the numbers to be used. The 0-complement is also an $LS(n)$ and it can be filled out with a solution using the n new values $[2n+1, 2n+2, \dots, 3n]$: in the figure, the 4 new values are denoted by $a\dots d$. Using Ryser's Theorem, we see that $\mathcal{M} = n$, so there exists an $LS(3n)$ -embedding of the original LT.

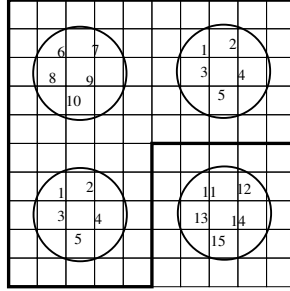


Figure 16: An $LT(N)$ with an $LS(N+N/2)$ -embedding. The circle with n numbers stands for an $LS(n)$ on those numbers - shown for $N=10$, but this works for any even N .

This embedding is minimal, because with $(n-1)$ new values, each occurring $(n+1)$ times (at least), there are not enough cells left in the 0-complement to have the values $(n+1)..2n$ occur the $(n+1)^{th}$ time as well.

6.3 LT(N) with gap $N/2+1$ for even N

[5] shows how to construct a (single) diagonal LS for $N \geq 3$. This will come in handy for $LT[2n(n), (n+1), n(n-1)]$ of which there is an example in Figure 17 for $n = 4$.

In Figure 17, the 0-complement plus the extra cell in the middle, is filled out with a diagonal LS solution. Then the diagonal is replaced by a new value. \mathcal{M} now equals $(n-1)$ and the number of new values is $(n+1)$. Ryser's Theorem guarantees an embedding in $LS(3n+1)$. There is no embedding in an $LS(3n)$: \mathcal{M} and the number of new values should not be smaller than n in that case, but the 0-complement has only $n^2 - 1$ cells.

It feels like this can be generalized to $LT(N)$ with larger and larger gaps.

				a	c	d	b
				d	b	a	c
				b	d	c	a
				c	a	b	d

					c	d	b
				d	e	a	c
				b	d	e	a
				c	a	b	e

Figure 17: $LT[8(4),5,4(3)]$ with its 0-complement filled out

6.4 $LT(N)$ with gap 3

Figure 18 can be generalized to all odd $N \geq 5$.

									a
								a	b
							a	b	c
							b	c	9
						7	c	9	8
			a	b	c	9	8	7	6
		a	b	c	9	8	7	6	5
	a	b	c	9	8	7	6	5	

Figure 18: An $LT(N)$ with an $LS(N+3)$ -embedding. Shown for $N=9$, but it works for any odd $N \geq 5$

This tableau is $LT[2n+1, 2n, \dots, n+2, n+2, n+1, n-1, n-2, \dots, 2, 1]$. The embedding in $LS(N+3)$ is minimal. Indeed, the 0-complement of this tableau is $LT[2n, 2n-1, \dots, n+2, n, n-1, n-2, \dots, 1]$, and the 2-complement is $LT[2n+3(2), 2n+2, \dots, n+5, n+4, n+4, n+2, n+1, \dots, 3, 2]$. Take the subpartition that starts at $(n+5)$, i.e. $[n+5, n+4, n+4, n+2, n+1, \dots, 3, 2]$. The conjugate starts with $[n+5, n+5, \dots]$ so this subpartition does not dominate its conjugate and the 2-complement is not wide, so it cannot have a solution.

6.5 Two more interesting classes of $LT(N)$

Below are two more classes of Latin Tableaux that showed promise to make hard claims about the existence of gaps in a certain range, but lack of time prevented us from pursuing this.

- We denote $LT[N(i), N-1, N-2, \dots, i]$ by $DLT(N, i)$. $DLT(N)$ is equal to $DLT(N, 1)$.

While the 0-complement of $DLT(N)$ is a $DLT(N-1)$, the 0-complement of a $DLT(N,i)$ is a $DLT(N-i)$.

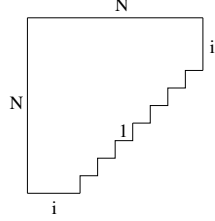


Figure 19: $DLT(N,i) = LT[N(i), N-1, N-2, \dots, i]$

This class shows all odd gaps.

- We denote $LT[N(i), (N-b)(N-i-b), i(b)]$ by $LTP[N,i,b]$ - see Figure 20. Clearly $LTP[N,i] = LTP[N,i,1]$, so $LTP[N,i,b]$ generalizes $LTP[N,i]$. $LTP[N,i,b]$ has a solution only if $b \leq N/2$ and $i \geq b$.

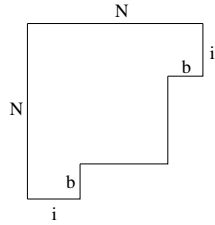


Figure 20: $LTP[N,i,b] = LT[N(i), (N-b)(N-i-b), i(b)]$

This class shows (among some others) all gaps in $[N/2 \dots N]$.

7 Conclusion and future work

Our initial interest in Latin Tableaux was related to redundant disequalities in the CSP formulation of $LT(N)$. We strayed into the study of embeddings of an $LT(N)$ in a potentially larger Latin Square. The following results were obtained: (1) an algorithm to extend any self-conjugate partition minimally to a wide one with the same shape (2) a proof that there exists a gap 2 $LT(N)$ if $(N+2)$ is composite and a strong conjecture that the condition is also necessary (3) the description of classes of LTs with particular minimal embeddings.

The first result uses *full column extension* to achieve wideness. It might be interesting to study what one can achieve with extensions that use partial columns. In particular the question *how can the shape be adapted minimally to obtain a wide (or solvable) LT* seems interesting.

The two other results indicate that there exists a gap M $\text{LT}(N)$ for all $3 \leq M \leq N$, that there exists a gap 2 $\text{LT}(N)$ if and only if $(N+2)$ is composite. Moreover, there exists only one gap 1 $\text{LT}(N)$ for all N .

However, some of our results depend (for now) on the truth of the Wide Partition Conjecture.

The study of the complexity of the completion decision problem in the context of Latin Tableaux is intriguing: for Latin Squares it is NP-complete (see [3]). For $\text{DLT}(N)$, the completion problem is in P (even with small exponent). In some sense, DLT 's are the smallest LTs and LSs are the largest LTs, so we expect that there is a border at which going from DLT to general LT, the decision problem transits from P to NP-complete. Finding characterizations of this border would be nice: in [7], a border between DTL and LS concerning redundancy of sets of disequalities was identified, but a complexity border would be even more interesting. However, at this moment, we don't even know whether the completion problem for the minimal deviation of $\text{DLT}(N)$, namely $\text{LT}_d(2d-2)$ is in P.

Several other topics might be worth studying: 3-dimensional Latin Tableaux (as also Latin Cubes have become popular), the minimal amount of clues needed to uniquely complete a partial Latin Tableau, its critical sets, the minimal number of disequality constraints to be violated for filling out a given (unsolvable) $\text{LT}(N)$, or alternatively the minimal domain extensions.

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